Lecture 2
Discrete-time Queues

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Outline

- Discrete-time Markov Chains
  - Reading: Section 3.3 of Srikant & Ying
- Little's Law
- The Geo/Geo/1 Queue
  - R. Srikant and Lei Ying, *Communication Networks: An Optimization Control and Stochastic Networks Perspective*, Cambridge University Press, 2014.

Discrete-time Markov Chains

- Let X_k be a discrete-time stochastic process that takes on values in a countable set. X_k is called a *Discrete-Time Markov Chain (DTMC)*, or simply Markov Chain, if

$$\Pr(X_k = i_k | X_{k-1} = i_{k-1}, X_{k-2} = i_{k-2}, \ldots) = \Pr(X_k = i_k | X_{k-1} = i_{k-1})$$

  - Given the present state, the future is independent of the past

- A Markov chain is called *time-homogenous*, if $\Pr(X_k = j | X_{k-1} = i)$ is independent of *k*

- We only consider time-homogenous Markov chains

Discrete-time Markov Chains (2)

- Each time-homogenous Markov chain has a transition probability matrix **P**, such that

$$P_{ij} = \Pr(X_k = j | X_{k-1} = i)$$

- Let *p[k]* denote a vector of probabilities with

$$p_j[k] = \Pr(X_k = j)$$

  - Then, $p[k] = p[k-1]\mathbf{P}$

Key Questions

The following questions are important in the study of Markov chains.

- Does there exist a distribution vector π so that $\pi = \pi \mathbf{P}$?
  - If it exists, it is called a stationary distribution.
- If there exists a unique stationary distribution, does convergence $\lim_{k \to \infty} p[k] = \pi$ hold for all $p[0]$?

Irreducible Markov Chains

Definition 3.3.1 Let $P_{ij}^{(n)} = \Pr(X_{k+n} = j \mid X_k = i)$.

(1) State j is said to be reachable from state i if there exists $n \geq 1$ so that $P_{ij}^{(n)} > 0$.
(2) A Markov chain is said to be *irreducible* if any state i is reachable from any other state j. □

- Exercise 1. two-state Markov chain with $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - Is it irreducible?
  - What is its stationary distribution?

Exercise 2

- Consider a two-state Markov chain with $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  - Is it irreducible?
  - What is its stationary distribution?
  - Does $\lim_{k \to \infty} p[k] = \pi$?

Aperiodic Markov Chains

Definition 3.3.2 The following definitions classify Markov chains and their states as periodic or aperiodic.

(1) State i is said to have a period $d_i \geq 1$ if $d_i = \gcd\left\{n : P_{ii}^{(n)} > 0\right\}$, where gcd denotes the greatest common divisor. If $P_{ii}^{(n)} = 0, \forall n$, we say that $d_i = \infty$.

(2) State i is said to be *aperiodic* if $d_i = 1$.

(3) A Markov chain is said to be *aperiodic* if all states are aperiodic. □

- The greatest common divisor of {4,6,8} is 2.
- The greatest common divisor of {3,7,9} is 1, where 3 and 7 are prime numbers.

Lemma 3.3.1 Every state in an irreducible Markov chain has the same period. Thus, in an irreducible Markov chain, if one state is aperiodic, the Markov chain is aperiodic. □

Finite State Space

Theorem 3.3.2 A finite-state-space, irreducible Markov chain has a unique stationary distribution π, and, if it is aperiodic, $\lim_{k\to\infty} p[k] = \pi$, $\forall p[0]$. □

- Finite state space + irreducible + aperiodic ➔ existence + uniqueness + convergence to stationary distribution

Infinite State Space

- If the state space is infinite, the existence of a stationary distribution is not guaranteed, even if the Markov chain is irreducible.

- Exercise 3. Markov chain with infinite state space and

$$\mathbf{P} = \begin{bmatrix} 2/3 & 1/3 & 0 & 0 & 0 & \dots \\ 1/3 & 1/3 & 1/3 & 0 & 0 & \ddots \\ 0 & 1/3 & 1/3 & 1/3 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

  - Is it irreducible?
  - Is it aperiodic?
  - What is its stationary distribution?

Positive Recurrent Markov Chains

- Additional property is needed for existence of stationary distributions in countable-state-space Markov chain

Definition 3.3.3 The following definitions classify the states of a Markov chain as recurrent or transient.

(1) The recurrence time T_i of state i of a Markov chain is defined as

$$T_i = \min\{n \geq 1 : X_n = i \text{ given } X_0 = i\}.$$

(Note that T_i is a random variable.)

(2) A state i is said to be *recurrent* if $\Pr(T_i < \infty) = 1$. Otherwise, it is called *transient*.

(3) The mean *recurrence time* M_i of state i is defined as $M_i = E[T_i]$.

(4) A recurrent state i is called *positive recurrent* if $M_i < \infty$. Otherwise, it is called *null recurrent*.

(5) A Markov chain is called *positive recurrent* if all of its states are positive recurrent. □

Positive Recurrent Markov Chains (2)

Lemma 3.3.3 Suppose $\{X_k\}$ is irreducible and that one of its states is positive recurrent, then all of its states are positive recurrent. (The same statement holds if we replace positive recurrent by null recurrent or transient.) $\square$

Lemma 3.3.4 If state i of a Markov chain is aperiodic, then $\lim_{k\to\infty} p_i[k] = 1/M_i$. (This is true whether or not $M_i < \infty$, and even for transient states by defining $M_i = \infty$ when state i is transient.) $\square$

Theorem 3.3.5 Consider a time-homogeneous Markov chain which is irreducible and aperiodic. Then, the following results hold.

- If the Markov chain is positive recurrent, there exists a unique π such that $\pi = \pi\mathbf{P}$ and $\lim_{k\to\infty} p[k] = \pi$. Further, $\pi_i = 1/M_i$.
- If there exists a positive vector π such that $\pi = \pi\mathbf{P}$ and $\sum_i \pi_i = 1$, it must be the stationary distribution and $\lim_{k\to\infty} p[k] = \pi$. (From Lemma 3.3.4, this also means that the Markov chain is positive recurrent.)
- If there exists a positive vector π such that $\pi = \pi\mathbf{P}$, and $\sum_i \pi_i$ is infinite, a stationary distribution does not exist, and $\lim_{k\to\infty} p_i[k] = 0$ for all i. $\square$

Computing Stationary Distributions

- Solve $\pi = \pi \mathbf{P}$, $\sum_i \pi_i = 1,\ \pi_i \geq 0$
- There are instances where one cannot easily solve the equation. But we would still like to know if the stationary distribution exists.
- It is often easy to verify the irreducibility and aperiodicity of a Markov chain, but, in general, it is difficult to verify directly whether a Markov chain is positive recurrent from the definitions

Sufficient Condition for Positive Recurrent

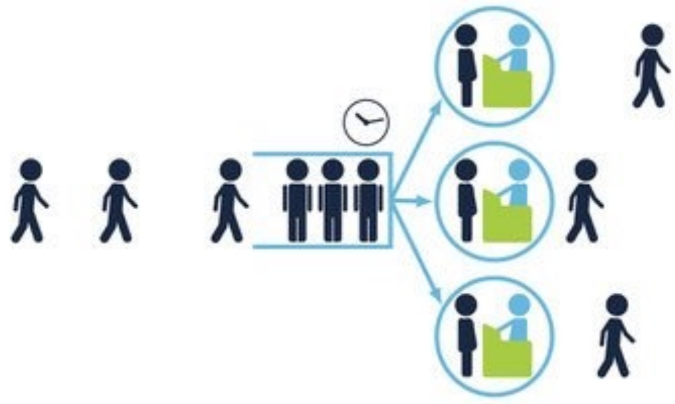
- There is a convenient test called Foster's test or the Foster–Lyapunov test to check for positive recurrence:

Theorem 3.3.7 (Foster–Lyapunov theorem) Let $\{X_k\}$ be an irreducible Markov chain with a state space $\mathcal{S}$. Suppose that there exist a function $V : \mathcal{S} \rightarrow \mathcal{R}^+$ and a finite set $\mathcal{B} \subseteq \mathcal{S}$ satisfying the following conditions:

(1) $E[V(X_{k+1}) - V(x) \mid X_k = x] \leq -\epsilon$ if $x \in \mathcal{B}^c$ for some $\epsilon > 0$, and
(2) $E[V(X_{k+1}) - V(x) \mid X_k = x] \leq A$ if $x \in \mathcal{B}$ for some $A < \infty$.

Then the Markov chain $\{X_k\}$ is positive recurrent.

Exercise 4. Stability of a Queue



- Consider a queue with $Q(t)$ customers at time-slot t. $a(t)$ customers arrive at the beginning of time-slot t, $b(t)$ customers leaves at the end of time-slot t. $Q(t)$ is a Markov chain, which evolves according to

$$Q(t+1) = \max\{Q(t) + a(t) - b(t), 0\}$$

$a(t)$ = 0 or 4 with prob. 0.5, and it is i.i.d. over time. $b(t) = \min\{Q(t), 3\}$.

- Question: Does $Q(t)$ converge to a unique stationary distribution?

Summary

- Key questions for Markov chain
  - Existence of stationary distribution ?
  - Convergence to unique stationary distribution ?

- Finite-state-space Markov chain
  - Irreducible + aperiodic → existence + uniqueness + convergence

- Countable-state-space Markov chain
  - Foster-Lyapunov → positive recurrent
  - Irreducible + aperiodic + positive recurrent → existence + uniqueness + convergence

- Reading: Section 3.3 of Srikant & Ying